

Existence of the solution to electromagnetic wave scattering problem for an impedance body of an arbitrary shape

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Abstract

A new proof is given of the existence of the solution to electromagnetic (EM) wave scattering problem for an impedance body of an arbitrary shape. The proof is based on the elliptic systems theory and elliptic estimates for the solutions of such systems.

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a connected smooth boundary S , $D' := \mathbb{R}^3 \setminus D$, $k^2 = \text{const} > 0$, $\omega > 0$ is frequency, $\zeta = \text{const}$, $\text{Re}\zeta \geq 0$, be the boundary impedance, $\epsilon > 0$ and $\mu > 0$ are dielectric and magnetic constants, $\epsilon' = \epsilon + i\frac{\sigma}{\omega}$, $\sigma = \text{const} \geq 0$, $x \in D'$, $r = |x|$, N is the unit normal to S pointing into D' .

Consider the problem

$$\nabla \times e = i\omega\mu h, \quad \nabla \times h = -i\omega\epsilon' h \quad \text{in } D', \quad (1)$$

$$r(e_r - ike) = o(1), \quad r \rightarrow \infty, \quad (2)$$

$$[N, [e, N]] - \frac{\zeta}{i\omega\mu}[N, \text{curl } e] = -f. \quad (3)$$

MSC: 78A45; 78A25

key words: electromagnetic wave scattering; small impedance body; scatterer of an arbitrary shape; complementary condition; elliptic systems

Here f is a given smooth tangential field to S , $[A, B] = A \times B$ is the cross product of two vectors, $A \cdot B$ is their scalar product. Problem (1)-(3), which we call *problem I*, is the scattering problem for electromagnetic (EM) waves for an impedance body D of an arbitrary shape. This problem has been discussed in many papers and books. Uniqueness of its solution has been proved (see, e.g., [3], pp.81-83). Existence of its solution was discussed much less (see [2], pp.254-256). Explicit formula for the plane EM wave scattered by a small impedance body ($ka \ll 1$, a is the characteristic size of this body) of an arbitrary shape is derived in [3]. There one can also find a solution to many-body scattering problem in the case of small impedance bodies of an arbitrary shape.

The aim of this paper is to outline a method for proving the existence of the solution to *problem I* based on elliptic theory and on a result from [4]. It is clear that *problem I* is equivalent to *problem II*, which consists of solving the equation

$$(\Delta + k^2)e = 0 \quad \text{in } D', \quad (4)$$

assuming that e satisfies conditions (2), (3) and

$$\text{Div } e = 0 \quad \text{on } S. \quad (5)$$

Conditions (5), (2), and equation (4) imply $\nabla \cdot e = 0$ in D' . If *problem II* has a solution e , then the pair $\{e, h\}$ solves *problem I*, provided that $h = \text{curl } e / (i\omega\mu)$. The solution to *problem II*, if it exists, is unique, because *problem I* has at most one solution and is equivalent to *problem II*. This solution satisfies the following estimate:

$$\|e\|^2 := \|e\|_0^2 := \int_{D'} |e(x)|^2 w(x) dx \leq c, \quad w(x) := (1 + |x|)^{-d}, \quad d = \text{const} > 1. \quad (6)$$

We denote by $H^m(D', w)$ the weighted Sobolev space with the weight w , by $\|e\|_m$ the norm in $H^m(D', w)$, and by $|e|_m$ the norm in $H^m(S)$, where $H^m(S)$ is the usual Sobolev space of the functions on S and m need not be an integer.

Let us outline the ideas of our proof.

Step 1. One checks that problem (5), (3), and equation (4) is an elliptic problem, i.e., equation (4) is elliptic (this is obvious) and the boundary conditions (5), (3), satisfy the Lopatinsky-Shapiro (LS) condition (see, e.g., [1] for the definition of LS condition which is also called ellipticity condition for the operator in (4) and the boundary conditions (5), (3), or the complementary condition, see also [6]).

Step 2. Reduction of *problem II* to the form from which it is clear that *problem II* is of Fredholm type and its index is zero.

Step 3. Derivation of the estimate:

$$|e|_{m+1} \leq c|f|_m, \quad m > 1/2, \quad (7)$$

where $\text{Re} \zeta > 0$, $c = \text{const} > 0$ does not depend on e or f .

Let us formulate our result.

Theorem 1. *For any tangential to S field $f \in H^m(S)$ problem II has a (unique) solution $e \in H^{m+(3/2)}(D', w)$, $e|_S \in H^{m+1}(S)$, and estimates (6) and (7) hold.*

In Section 2 we prove Theorem 1.

2 Proof of Theorem 1

Proof of Theorem 1. Step 1. The principal symbol of the operator in (4) is $\xi^2 \delta_{pq}$, δ_{pq} is the Kronecker delta, $\xi^2 = \sum_{j=1}^3 \xi_j^2$, so system (4) is elliptic. Let us rewrite (4) and boundary conditions (3) and (5) as follows:

$$P(D)e = (D_1^2 + D_2^2 + D_3^2 - k^2)e = 0 \quad \text{in } D', \quad (8)$$

$$B(D)e := \left\{ \frac{\zeta}{i\omega\mu} [N, \text{curl } e] - [N, [e, N]] = f, \quad \sum_{p=1}^3 D_p e_p = 0 \right\} \quad \text{on } S, \quad (9)$$

where $D_j = -i\partial/\partial x_j$ and $D = (D_1, D_2, D_3)$. The principal part of (8), which defines its principal symbol, is

$$P'(D) = D_1^2 + D_2^2 + D_3^2, \quad (10)$$

where the prime in $P'(D)$ denotes the principal part of (8). If we take the local coordinate system in which $N = (0, 0, 1)$, then the principal part of the boundary operator (9) is the matrix

$$B'(D) := \frac{\zeta}{i\omega\mu} \begin{pmatrix} -D_3 & 0 & D_1 \\ 0 & -D_3 & D_2 \\ D_1 & D_2 & D_3 \end{pmatrix} \quad (11)$$

and its symbol is

$$B'(\xi) = \frac{\zeta}{i\omega\mu} \begin{pmatrix} -i\xi_3 & 0 & i\xi_1 \\ 0 & -i\xi_3 & i\xi_2 \\ i\xi_1 & i\xi_2 & i\xi_3 \end{pmatrix} \quad (12)$$

The operator $\frac{\partial}{\partial x_p}$ is mapped onto $i\xi_p$. The principal symbol of the operators in the boundary conditions (3),(5) is calculated in the local coordinates in which x_3 -axis is directed along N . The third row in matrix (12) corresponds to condition (5). The first two rows correspond to the expression $[N, \text{curl } e] := (\text{curl } e)_\tau$, which is responsible for the principal symbol corresponding to boundary condition (3).

To check if the LS condition is satisfied, we must show that the only rapidly (exponentially) decreasing solution of the problem

$$P'(\xi_1, \xi_2, D_t)u(\xi, t) = 0, \quad t > 0, \quad (13)$$

$$B'(\xi_1, \xi_2, D_t)u(\xi, 0) = 0, \quad (14)$$

is the zero solution. Here $D_t = -i\partial/\partial t$.

The set of rapidly decreasing solutions to the equation, corresponding to the principal symbol of (4), is $\{v_m e^{-t\rho}\}$, where $\rho := (\xi_1^2 + \xi_2^2)^{1/2}$ and vectors v_m are linearly independent. Thus, if $\xi' := \{\xi_1, \xi_2\}$ and $u(\xi', t)$ is a rapidly decreasing solution of (4), then $D_t u(\xi', t) = i\rho u(\xi', t)$. Therefore,

$$B'(\xi', D_t)u(\xi', t) = \frac{\zeta}{i\omega\mu} \begin{pmatrix} -D_t & 0 & i\xi_1 \\ 0 & -D_t & i\xi_2 \\ i\xi_1 & i\xi_2 & D_t \end{pmatrix} u(\xi', t) = \frac{\zeta}{i\omega\mu} \begin{pmatrix} -i\rho & 0 & i\xi_1 \\ 0 & -i\rho & i\xi_2 \\ i\xi_1 & i\xi_2 & i\rho \end{pmatrix} u(\xi', t). \quad (15)$$

The LS condition holds if the matrix

$$\begin{pmatrix} -i\rho & 0 & i\xi_1 \\ 0 & -i\rho & i\xi_2 \\ i\xi_1 & i\xi_2 & i\rho \end{pmatrix} \quad (16)$$

is non-singular for all $\xi' \neq 0$. The determinant of this matrix is $-2i\rho^3 \neq 0$ for $\rho > 0$. Therefore, the LS condition holds. \square

Step 2 and Step 3. To check that $\kappa = 0$, where κ is the index of *problem II*, let us transform this problem using the result in [4], where it is proved that for $\zeta = 0$ the solution e to *problem II* exists, e is uniquely determined by f , and e has the same smoothness as f . This follows from the results in [4] under the assumption that the domain D is small, which implies that k^2 is not an eigenvalue of the Dirichlet Laplacean in D . If D is not small then this result follows from the fact that the relation $\text{curl} \int_{D'} g(x, t) J(t) dt = 0$ in D' implies $J = 0$ on S if J is a tangential to S field. In proving this one assumes that k^2 is not an eigenvalue of the Dirichlet Laplacean in D . This is not an essential restriction: see [5], p.20, Section 1.3. The map $V : f \rightarrow e_\tau$, where e_τ is the tangential to S component of e , acts from $H^m(S)$ onto $H^m(S)$, and V is an isomorphism of $H^m(S)$ onto itself.

Rewrite equation (3) as

$$e_\tau = -Vf + \frac{\zeta}{i\omega\mu} V((\text{curl } e)_\tau). \quad (17)$$

The operator $V((\text{curl } e)_\tau)$ preserves the smoothness of $(\text{curl } e)_\tau$, and so, if $\text{Re}\zeta > 0$, it acts from $H^m(S)$ into $H^{m+1}(S)$ due to equation (17). Indeed, if e_τ belongs to $H^{m+1}(S)$ then $(\text{curl } e)_\tau$ belongs to $H^m(S)$. On the other hand, equation (17) implies that the smoothness of $V((\text{curl } e)_\tau)$ is not less than the smoothness of e_τ , which is $H^{m+1}(S)$. Also, equation (17) implies that the smoothness of e_τ is not less than the smoothness of Vf , which is $H^{m+1}(S)$ if $f \in H^{m+1}(S)$. Therefore, if $\text{Re}\zeta > 0$ then V acts from $H^m(S)$ into $H^{m+1}(S)$, so it is compact in $H^m(S)$, and equation (17) is of Fredholm type with index zero. Knowing e_τ on S one can uniquely recover e in D' .

Since problem (8)- (9) is an elliptic system obeying the LS condition, we have the elliptic estimate (see, e.g., [6]):

$$\|e\|_m \leq C(\|P(D)e\|_{m-2} + |B(D)e|_{m-(3/2)} + \|\psi e\|_0), \quad e \in H^m(D', w), \quad (18)$$

where $m > 3/2$ and $\psi \in C_0^\infty(\mathbb{R}^3)$. Recall that $P(D)e = 0$. Hence, a solution of (8), (9) satisfies

$$\|e\|_m \leq C(|f|_{m-(3/2)} + \|\psi e\|_0). \quad (19)$$

Equation (17) has at most one solution if $\text{Re}\zeta \geq 0$ because *problem II* has at most one solution. Therefore, by the Fredholm alternative, equation (17) has a solution, this solution is unique, and estimate (7) holds due to ellipticity of the *problem II*.

Estimate (6) holds because $e = O(1/r)$ when $r \rightarrow \infty$.

We now want to prove that e belongs to $H^m(D', w)$ where m is determined by the smoothness of f .

Lemma 1. *The following estimate holds for a solution to problem II with $\operatorname{Re}\zeta > 0$:*

$$\|e\|_m \leq c|f|_{m-(3/2)}. \quad (20)$$

Here and below $c > 0$ stand for various estimation constants.

Proof. If (20) is false, then there is a sequence of f_n such that

$$\|e_n\|_m \geq n|f_n|_{m-(3/2)}, \quad \|e_n\|_m = 1. \quad (21)$$

Thus, there is a subsequence denoted again e_n and an $e \in H^m(D', w)$ such that $e_n \rightarrow e$ weakly in $H^m(D', w)$, strongly in $H^{m'}(D', w)$, $m' < m$, and almost everywhere in D' . Since $\psi \in C_0^\infty(\mathbb{R}^3)$, we have $\psi e_n \rightarrow \psi e$ strongly in $L^2(D')$. Hence by (19),

$$\|e_j - e_k\|_m \leq C(|f_j - f_k|_{m-(3/2)} + \|\psi e_j - \psi e_k\|_0) \rightarrow 0 \quad (22)$$

as $j, k \rightarrow \infty$. Thus, $e_n \rightarrow e$ in $H^m(D', w)$, so that $\|e\|_m = 1$ while $f = 0$. Consequently, e solves *problem II* with $f = 0$, so $e = 0$. This contradicts the fact that $\|e\|_m = 1$.

Let us give an alternative proof of the convergence of e_n to e in $H^m(D', w)$.

One has, by Green's formula,

$$e_n(x) = \int_S \left(e_n(t)g_N(x, t) - g(x, t)(e_n)_N(t) \right) dt. \quad (23)$$

Pass to the limit $n \rightarrow \infty$ in this formula, use convergence $|e_n - e|_{m-(3/2)} \rightarrow 0$ and $|(e_n)_N - e_N|_{m-(3/2)} \rightarrow 0$ and get

$$e(x) = \int_S \left(e(t)g_N(x, t) - g(x, t)e_N(t) \right) dt. \quad (24)$$

This equation implies that e solves equation (4) and satisfies the radiation condition (2). Furthermore, it satisfies equation $\nabla \cdot e = 0$ in D' because one can pass to the limit $n \rightarrow \infty$ in equation (5) for e_n , and if $\operatorname{Div} e = 0$ on S then $\nabla \cdot e = 0$ in D' . Indeed, $\nabla \cdot e$ satisfies equation (4) and the radiation condition (2), so if it vanishes on S then it vanishes in D' . Moreover, e satisfies equation (3) with $f = 0$ because $f_n \rightarrow 0$ as $n \rightarrow \infty$. By the uniqueness theorem, $e = 0$ in D' .

Let us check that $\|e_n - e\|_m \rightarrow 0$. Locally this convergence is already checked, so one has to check convergence in the weighted norm near infinity. Estimate (23) implies

$$|e_n(x)| \leq c|x|^{-1}, \quad (25)$$

where the constant $c > 0$ does not depend on n because of the convergence $|e_n - e|_{m-0.5} \rightarrow 0$ and $|(e_n)_N - e_N|_{m-1.5} \rightarrow 0$. Estimate (25) implies the desired convergence near infinity in the weighted norm because of the assumption $d > 1$. Therefore, we have a contradiction: $\|e_n\|_m = 1$ and $\|e_n - e\|_m = \|e_n\|_m \rightarrow 0$. This contradiction proves Lemma 1. \square

This completes the outline of the proof of Theorem 1. \square

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